Hydromagnetic waves in a cylindrical plasma

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This paper contained several extensions of the theory of hydromagnetic waves in a partially ionized gas. The gas is confined in a cylindrical tube through which passes an axial magnetic field. The tube wall is assumed to be either a perfect conductor or a material of small or zero conductivity.

The effects of the viscosity and compressibility of the ionized and neutral gases are included in the theory, as also are the contributions of finite conductivity and the ion-cyclotron term. The non-isotropic character of the viscosity and conductivity coefficients of the ionized gas is taken into account. New boundary conditions are derived for tubes with insulating walls, and it is shown that the only pure modes that can be propagated along such tubes are just two special cases of those waves having no azimuthal dependence.

The principal result of the paper is a new dispersion relation which allows for all the dissipative effects just mentioned and which is valid for a range of frequencies which extends well beyond the ion cyclotron frequency, but falls short of the frequency at which electron inertia and displacement currents became effective.

1. Introduction

The hydromagnetic oscillations of a cylindrical plasma have been investigated by a number of authors (Newcomb 1957; Stix 1957, 1958; Lehnert 1959; Gajewski 1959; and many others). Both Newcomb (1957) and Stix (1957) have investigated the effect of retaining the ion-cyclotron term in the generalized Ohm's law and Piddington (1956) and Lehnert (1959) have allowed for the presence of neutral gas in the plasma. However, there does not seem to be general treatment of the problem in which all the dissipative and other second-order effects are present and in which the plasma is confined to a cylindrical tube. This paper does not deal with the most general case for it is assumed that conditions are such that the displacement current can be neglected in the plasma, but apart from this limitation it is more general than previous work.

The extent to which the theory given below is valid at frequencies at or near the ion-cyclotron frequency ω_{ci} is uncertain for in this region the usual expansion method of obtaining the transport coefficients from Boltzmann's equation may break down. However, to the extent to which viscosity and resistivity can be ignored, the results are formally correct at $\omega = \omega_{ci}$. The presence of the neutral gas, which eliminates the sharp resonance at the ion-cyclotron frequency, improves the range of validity of the theory.

Present experiments (see Jephcott 1959 and Jephcott, Stocker & Woods 1961) at A.E.R.E. (Harwell, England), involve discharge tubes with insulating rather than the conducting walls used in most of the American experiments (see Allen, Baker, Pyle & Wilcox 1959 and Wilcox, Boley & De Silva (1959). With insulating walls and negligible plasma resistivity there is some difficulty in satisfying the boundary conditions. The method chosen in § 9 is to restrict the type of perturbation to either purely torsional or purely radial motion. It appears that other types of wave are not possible, unless a current sheet can form near the insulating wall.

The main contribution of this paper is a very general dispersion relation (equation (27)) in which allowance is made for the non-isotropic character of the viscosity as well as for the compressibility of both the ionized and neutral gases. The manner in which the dissipative terms affect the waves at frequencies close to the ion-cyclotron frequency is also investigated. The dispersion relation is rather involved algebraically, and so it was found desirable to calculate the various relations between wave velocity, frequency and the damping terms on a digital computor. These results will be presented and discussed later in a Culham Laboratory report. In a further paper (Jephcott & Stocker 1962), computed solutions of the dispersion relation are compared with experimental results obtained at A.E.R.E., Harwell; the theory is found to be in good agreement with experiment.

2. Nomenclature

The standard electromagnetic symbols $\mathbf{B}$, $\mathbf{j}$, $\mathbf{E}$, μ, and σ, denoting magnetic induction, current density, electric field, inductive capacity and electrical conductivity, and also the standard gas-dynamics symbols $\mathbf{v}$, p, ρ, ν, and C denoting velocity, scalar pressure, density, kinematic viscosity and sound speed, all in rationalized M.K.S. units, are used throughout the paper. The vorticity vector $\nabla \times \mathbf{v}$ will be denoted by $\boldsymbol{\zeta}$.

A subscript n on the gas dynamics symbols is used to distinguish neutral gas quantities, while the subscripts $_\perp$ and $_\parallel$ are attached to the ionized gas transport coefficients ν and σ to denote values perpendicular and parallel to the steady magnetic field. The subscripts 0 and 1, when attached to dependent variables, denote steady and perturbation values respectively (see equation (1)).

The wave frequency is $\omega/2\pi$. The effective collision frequency between ions and neutrals is written as $2\omega_{in}$ for convenience, and it appears in the theory in the combinations $\lambda \equiv \omega_{in}/\omega$ and $\xi \equiv (\rho_{no}/\rho_0)(\omega/\omega_{in})$. The ion-cyclotron frequency ω_{ci} and the Alfven speed $v_A = B_0/\surd(\mu\rho_0)^{\frac{1}{2}}$ will emerge in the ratios $\Omega \equiv \omega/\omega_{ci}$ and $k_A \equiv \omega/v_A$, the sound speeds in the ratios $\Gamma \equiv C^2/\omega^2$ and $\Gamma_n \equiv C_n^2/\omega^2$, and the viscosities and resistivities in the ratios $\gamma \equiv \nu/\omega$ and $\delta \equiv 1/\mu\sigma\omega$. We shall also find it convenient to introduce the numbers $\gamma' = \gamma_\perp - \gamma_\parallel$ and $\delta' = \delta_\perp - \delta_\parallel$.

Finally, $\mathbf{n}$ will be used to denote a unit vector parallel to the steady magnetic field $\mathbf{B}_0$ along Oz, the axis of the plasma cylinder.

It will be assumed that the plasma oscillates about an equilibrium position such that a typical dependent variable, $\mathbf{A}$ say, can be expressed in the form

$$\mathbf{A}(r,\theta,z,t) = \mathbf{A}_0 + \mathbf{A}_1(r)\exp\{i(m\theta + kz - \omega t)\}, \tag{1}$$

where $\mathbf{A}_0$ is the equilibrium value of $\mathbf{A}$, m is an integer, k is the (complex) propagation number and r, θ, z are cylindrical co-ordinates.

3. Basic equations

Subject to the conditions described below, the equations for a partially ionized, viscous, compressible gas can be written (e.g. see Spitzer 1956 and Lehnert 1959)

$$\nabla \times \mathbf{B} = \mu \mathbf{j}, \tag{2}$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t, \tag{3}$$

$$(\partial/\partial t + \mathbf{v}.\nabla)\, p = -C^2 \rho \nabla . \mathbf{v}, \quad (\partial/\partial t + \mathbf{v}_n.\nabla)\, p_n = -C_n^2 \rho_n \nabla . \mathbf{v}_n, \tag{4}$$

$$\left.\begin{aligned} \rho(\partial/\partial t + \mathbf{v}.\nabla)\,\mathbf{v} &= -\nabla p + \mathbf{j} \times \mathbf{B} - \rho\omega_{in}(\mathbf{v}-\mathbf{v}_n) + \rho \boldsymbol{\nu}.(\nabla^2 \mathbf{v} + \tfrac{1}{3}\nabla\nabla . \mathbf{v}), \\ \rho_n(\partial/\partial t + \mathbf{v}_n.\nabla)\,\mathbf{v}_n &= -\nabla p_n + \rho\omega_{in}(\mathbf{v}-\mathbf{v}_n) + \rho_n \nu_n(\nabla^2 \mathbf{v}_n + \tfrac{1}{3}\nabla\nabla . \mathbf{v}_n), \end{aligned}\right\} \tag{5}$$

$$\mathbf{j} = \boldsymbol{\sigma}.\{\mathbf{E} + \mathbf{v}\times\mathbf{B} - (B_0/\omega_{ci})(\partial/\partial t + \mathbf{v}.\nabla)\,\mathbf{v} - (B_0/\rho\omega_{ci})\nabla . \mathbf{p}_i - B_0(\omega_{in}/\omega_{ci})(\mathbf{v}-\mathbf{v}_n), \tag{6}$$

$$\mathbf{p}_i = g\rho \mathbf{I} - \rho\boldsymbol{\nu}.(\nabla\mathbf{v} + \tfrac{1}{3}\mathbf{I}\nabla . \mathbf{v}), \tag{7}$$

$$\left.\begin{aligned} \boldsymbol{\sigma} &= \sigma_\perp(\mathbf{I} - \mathbf{nn}) + \sigma_\parallel \mathbf{nn} = \sigma_\perp \mathbf{I} + (\sigma_\parallel - \sigma_\perp)\,\mathbf{nn}, \\ \boldsymbol{\nu} &= \nu_\perp(\mathbf{I} - \mathbf{nn}) + \nu_\parallel \mathbf{nn} = \nu_\perp \mathbf{I} + (\nu_\parallel - \nu_\perp)\,\mathbf{nn}, \end{aligned}\right\} \tag{8}$$

where $\mathbf{I}$ is the idem tensor, g is the ratio of the ion temperature to the sum of the ion and electron temperatures, and $\mathbf{p}_i$ is the tensor pressure due to the ions alone.

In addition to the basic assumption that the plasma can be treated as a fluid, the most important assumptions implicit in these equations are that (i) the electron-ion collision frequency is large compared with the frequency ω defined in equation (1)—this permits us to drop the derivative of $\mathbf{j}$ from the generalized Ohm's law, equation (6); (ii) the ratio of the electron mass to the ion mass can be neglected compared with unity; (iii) the momentum transfer between the neutral and ionized gases occurs wholly in ion-neutral collisions; (iv) the electron viscosity is negligible compared with the ion viscosity (see, for example, Marshall 1960); and (v) the Alfvèn speed is small compared with the speed of light, which permits us to drop the displacement current from (2). Lehnert (1959) has given equations for a partially ionized gas in which assumptions (i) to (iii) are not made, although he makes no allowances for viscosity. In particular to eliminate (iii) it is necessary to introduce terms $\beta\mathbf{j}$, $-\beta\mathbf{j}$ into equations (5), where

$$\beta = (m_e/e)\,(\omega_{en} - \omega_{in}),$$

m_e is the electron mass and e its charge. However, it is apparent from the rows labelled β in tables 4 and 5 of Lehnert's paper that this term will contribute very little to the interaction.

The electrical conductivity and kinematic viscosity of the ionized gas have been assumed to be tensors to allow for the different values these numbers have in the transverse and magnetic field directions. Writing (6) in the form $\mathbf{j} = \boldsymbol{\sigma}.\mathbf{A}$ and using (8) we have

$$\mathbf{j} = \sigma_\perp \mathbf{A} + (\sigma_\parallel - \sigma_\perp)\,\mathbf{n}A_z, \quad j_z = \sigma_\parallel A_z,$$

thus

$$\mathbf{j} = \{(\sigma_\parallel - \sigma_\perp)/\sigma_\parallel\}\, j_z \mathbf{n} + \sigma_\perp \mathbf{A}, \tag{9}$$

a form we shall use below in place of (6).

We are interested in calculating the small perturbations from the equilibrium state $\mathbf{B}_0 = B_0\mathbf{n}, \quad \mathbf{j}_0 = \mathbf{v}_0 = \mathbf{v}_{n0} = 0, \quad \rho_{n0}, \rho_0 \quad \text{and} \quad p_0, p_{n0}$ constant, (10)

caused by the waves having the form indicated in equation (1). This form enables us to write $\mathbf{n}.\nabla = ik$, so that, for example,

$$(\nabla \times \mathbf{B}_1) \times \mathbf{n} = \mathbf{n}.\nabla\mathbf{B}_1 - \nabla(\mathbf{n}.\mathbf{B}_1) = ik\mathbf{B}_1 - \nabla B_{1z},$$

a type of reduction frequently used below.

On ignoring second-order terms in the perturbations, we find from (1) to (10) that the equations for the perturbations can be written

$$\nabla \times \mathbf{B}_1 = \mu\mathbf{j}_1, \tag{11}$$

$$\nabla \times \mathbf{E}_1 = i\omega\mathbf{B}_1, \tag{12}$$

$$\begin{aligned}\mathbf{j}_1 = \mu\omega\sigma_\perp\delta' j_{1z}\mathbf{n} + \sigma_\perp\{\mathbf{E}_1 + B_0\mathbf{v}_1 \times \mathbf{n} \\ + iB_0\Omega(1+i\lambda)\,\mathbf{v}_1 + B_0\lambda\Omega\mathbf{v}_{n1} - (B_0/\rho_0\omega_{ci})\,\nabla\,.\,\mathbf{p}_{i1}\},\end{aligned} \tag{13}$$

$$\begin{aligned}(1+i\lambda - i\gamma_\perp\nabla^2)\,\mathbf{v}_1 + i\gamma'\mathbf{n}\nabla^2 v_{1z} - \tfrac{1}{3}\gamma' k\mathbf{n}\nabla\,.\,\mathbf{v}_1 + (\Gamma - \tfrac{1}{3}i\gamma_\perp)\,\nabla\nabla\,.\,\mathbf{v}_1 \\ = i\lambda\mathbf{v}_{n1} - (\omega k/B_0 k_A^2)\,\mathbf{B}_1 - (i\omega/B_0 k_A^2)\,\nabla B_{1z},\end{aligned} \tag{14}$$

$$\mathbf{v}_1 = (Q - ia\nabla\nabla\,.\,)\,\mathbf{v}_{n1}, \tag{15}$$

where
$$a \equiv \xi(\Gamma_n - \tfrac{1}{3}i\gamma_n), \quad Q \equiv 1 - i\xi - \xi\gamma_n\nabla^2, \tag{16}$$

and we have taken advantage of (11) and the linearized forms of (4) to eliminate $\mathbf{j}_1$, p_1 and p_{n1}.

4. The dispersion relation

Relations (11) to (15) are five vector equations for the five unknowns $\mathbf{B}_1$, $\mathbf{v}_1$, $\mathbf{E}_1$, $\mathbf{j}_1$ and $\mathbf{v}_{n1}$. The term $\nabla\,.\,\mathbf{p}_{i1}$ in (13) can be expressed in terms of $\mathbf{v}_1$ and g by the linearized forms of (4) and (7); thus, provided we know the temperature ratio g, we have sufficient equations to solve our perturbation problem. The boundary conditions will be discussed later.

The method of solution which appears to involve least algebra is to use (12) and (15) to eliminate $\mathbf{E}_1$ and $\mathbf{v}_{n1}$, and then by some further differentiation reduce the remaining equations to four scalar equations relating the axial components B_{1z}, j_{1z}, v_{1z}, and ζ_{1z}. As $j_{1z} = (\mathbf{n}.\nabla \times \mathbf{B}_1)/\mu$, and $\zeta_{1z} = \mathbf{n}.\nabla \times \mathbf{v}_1$, we see that this choice has a kind of symmetry between the pairs of electrical and fluid variables. Further differentiation enables us to reduce the four scalar equations to a single differential equation for any one of the four axial components—they all satisfy the same complicated differential equation. The required dispersion relation then follows on showing that the equation is satisfied by a Bessel function.

First, we shall use (14) and (15) to derive a relation between v_{1z} and B_{1z}. On eliminating $\mathbf{v}_{n1}$ by (15) we get an equation which can be written in the form

$$\begin{aligned}P\mathbf{v}_1 + R\,.\,\nabla\nabla\mathbf{v}_1 + i\gamma'\mathbf{n}Q\nabla^2 v_{1z} + ia\gamma' k\nabla^2\nabla v_{1z} - \tfrac{1}{3}\gamma' k\mathbf{n}Q\nabla\,.\,\mathbf{v}_1 \\ = -(\omega k/B_0 k_A^2)\,Q\mathbf{B}_1 - (i\omega/B_0 k_A^2)\,(Q - ia\nabla^2)\,\nabla B_{1z},\end{aligned} \tag{17}$$

where
$$\left.\begin{aligned}P &\equiv Q + i\lambda(Q-1) - i\gamma_\perp Q\nabla^2, \\ R &\equiv (\Gamma - \tfrac{1}{3}i\gamma_\perp)\,Q - ia(1+\Gamma\nabla^2) - \tfrac{4}{3}a\gamma_\perp\nabla^2 + a\lambda - \tfrac{1}{3}a\gamma' k^2.\end{aligned}\right\} \tag{18}$$

The scalar product of (17) with $\mathbf{n}$ gives

$$\{P+i\gamma'(Q+iak^2)\nabla^2\}v_{1z}+ik\{R+\tfrac{1}{3}i\gamma'Q\}\nabla.\mathbf{v}_1 = -ia(\omega k_0/B_0k_A^2)\nabla^2B_{1z},$$

while its divergence is

$$\{P+R\nabla^2-\tfrac{1}{3}ik^2\gamma'Q\}\nabla.\mathbf{v}_1-k\gamma'\nabla^2(Q-ia\nabla^2)v_{1z} = -(i\omega/B_0k_A^2)(Q-ia\nabla^2)\nabla^2B_{1z},$$

and on eliminating first $\nabla.\mathbf{v}_1$ and then B_{1z} from these two equations we arrive at

$$B_0k_A^2Lv_{1z} = -\omega kM\nabla^2B_{1z} \tag{19}$$

and

$$kM\nabla.\mathbf{v}_1 = iNv_{1z}, \tag{20}$$

where

$$\left.\begin{aligned} L &\equiv \{P+i\gamma'(Q+iak^2)\nabla^2\}\{P+R\nabla^2-\tfrac{1}{3}ik^2\gamma'Q\} \\ &\qquad +ik^2\gamma'\{R+\tfrac{1}{3}i\gamma'Q\}\{Q-ia\nabla^2\}\nabla^2, \\ M &\equiv \{R+\tfrac{1}{3}i\gamma'Q\}\{Q-ia\nabla^2\}+ia\{P+R\nabla^2-\tfrac{1}{3}ik^2\gamma'Q\}, \\ N &\equiv \{P+i\gamma'Q\nabla^2\}\{Q-ia\nabla^2\}. \end{aligned}\right\} \tag{21}$$

Equation (19) is the required relation between v_{1z} and B_{1z}, and (20) will be used below to eliminate $\nabla.\mathbf{v}_1$.

The second of the four scalar equations relating axial components also follows from (17). From the z-component of the curl of (17) we get

$$B_0k_A^2P\zeta_{1z} = -\omega k\mu Qj_{1z}. \tag{22}$$

To find the third scalar equation we first eliminate $\mathbf{j}_1$, $\mathbf{E}_1$ and $\mathbf{v}_{n1}$ from (13) by operating on it with $\nabla\times(Q-ia\nabla\nabla.) = Q\nabla\times$ and then using (11), (12) and (15). The resulting equation involves $\nabla.\mathbf{v}_1$, which can be eliminated by (20), and $\nabla\times\nabla.\mathbf{p}_{i1}$, which can be eliminated by the following relation. The divergence of the linearized form of (7) is

$$\nabla.\mathbf{p}_{i1} = \nabla(gp_1)-\rho_0\boldsymbol{\nu}.(\nabla^2\mathbf{v}_1+\tfrac{1}{3}\nabla\nabla.\mathbf{v}_1),$$

hence

$$\nabla\times\nabla.\mathbf{p}_{i1} = -\rho_0\{\nu_\perp\nabla^2\boldsymbol{\zeta}_1-(\nu_\parallel-\nu_\perp)\mathbf{n}\times\nabla(\nabla^2v_{1z}+\tfrac{1}{3}ik\nabla.\mathbf{v}_1)\}.$$

The result of these operations is

$$\begin{aligned} k\omega Q(1-i\delta_\perp\nabla^2)M\mathbf{B}_1+i\mu\omega\sigma\delta'kMQ\mathbf{n}\times\nabla j_{1z}+B_0k^2QM\mathbf{v}_1-B_0QN\mathbf{n}v_{1z} \\ = -kB_0\Omega\{MP\boldsymbol{\zeta}_1-i\gamma'Q(M\nabla^2-\tfrac{1}{3}N)\mathbf{n}\times\nabla v_{1z}\}. \end{aligned} \tag{23}$$

If we now use (19) to eliminate B_{1z} from the axial component of (23) we get our third scalar equation, viz.

$$Q\{(k^2M-N)\nabla^2-k_A^2L(1-i\delta_\perp\nabla^2)\}v_{1z} = -k\Omega PM\nabla^2\zeta_{1z}. \tag{24}$$

The final equation is obtained by eliminating $\nabla.\mathbf{v}_1$ and j_{1z} by (20) and (22) from the z-component of the curl of (23). In this calculation it is necessary to make use of the result $\mathbf{n}.\nabla\times\boldsymbol{\zeta}_1 = \mathbf{n}.\nabla\times(\nabla\times\mathbf{v}_1) = \mathbf{n}.\nabla\nabla.\mathbf{v}_1-\nabla^2v_{1z}$. We find that

$$k\Omega QSv_{1z} = \{k^2Q-k_A^2P(1+i\delta'k^2-i\delta_\parallel\nabla^2)\}M\zeta_{1z}, \tag{25}$$

where

$$S \equiv P(P+R\nabla^2)+i\gamma'\{\tfrac{4}{3}P\nabla^2(Q-ia\nabla^2-\tfrac{1}{4}iak^2)+(M\nabla^2-\tfrac{1}{3}N)(\nabla^2+k^2)\}.$$

If this equation is used to eliminate ζ_{1z} from (24) and (25), there results $FQv_{1z} = 0$, where

$$\{k^2Q - k \equiv F_A^2 P(1 + i\delta' k^2 - i\delta_\parallel \nabla^2)\}\{k_A^2 L(1 - i\delta_\perp \nabla^2) + (N - k^2 M)\nabla^2\} - k^2\Omega^2 PS\nabla^2.$$

If either $Qv_{1z} = 0$, or $Fv_{1z} = 0$, the differential equation for v_{1z} will be satisfied, but from (16) it follows that the former cannot be generally true, so that the simplest differential equation for v_{1z} is

$$Fv_{1z} = 0. \tag{26}$$

It follows from (19), (22) and (24) that j_{1z}, B_{1z} and ζ_{1z} also satisfy this (sixteenth-order) differential equation.

One method of finding solutions of (26) is to assume that $\nabla^2\chi = \alpha\chi$ where χ is one of v_{1z}, j_{1z}, B_{1z} and ζ_{1z}, then this leads to a solution provided α is one of the roots of

$$\{k^2Q_\alpha - k_A^2 P_\alpha(1 + i\delta' k^2 - i\delta_\parallel \alpha)\}\{k_A^2 L_\alpha(1 - i\delta_\perp \alpha) + (N_\alpha - k^2 M_\alpha)\alpha\} = k^2\Omega^2 P_\alpha S_\alpha \alpha, \tag{27}$$

where P_α, Q_α, ..., etc., denotes the result of replacing ∇^2 by α in the operators $P, Q, \ldots$.

To solve $\nabla^2\chi - \alpha\chi = 0$ in a convenient form we set

$$-\alpha = k_c^2 + k^2, \tag{28}$$

and use (1) to find that

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\chi}{dr}\right) + \left(k_c^2 - \frac{m^2}{r^2}\right)\chi = 0; \tag{29}$$

hence

$$\chi = AJ_m(k_c r),$$

retaining only that solution which is finite at the origin. Let α_i, $i = 1, 2, \ldots, 8$ be the roots of (27) and k_{ci} be the corresponding values of the constant k_c defined in (28), then a solution of (26) is

$$\chi = \sum_{i=1}^{8} A_i J_m(k_{ci} r),$$

where A_i are constants. The constant k_c introduced in (28) depends on the boundary conditions. On eliminating α from (27) and (28) we arrive at the dispersion equation which will be investigated in some detail in § 8.

The above theory enables us to write

$$\left.\begin{aligned} B_{1z} &= k_c \mathscr{A} J_m(k_c r), & v_{1z} &= \Gamma k_c \mathscr{B} J_m(k_c r), \\ \mu j_{1z} &= k_c \mathscr{C} J_m(k_c r), & \zeta_{1z} &= k_c \mathscr{D} J_m(k_c r), \end{aligned}\right\} \tag{30}$$

where $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$ are constants related by

$$\left.\begin{aligned} \omega k M_\alpha \alpha \mathscr{A} &= -B_0 k_A^2 \Gamma L_\alpha \mathscr{B}, \quad B_0 k_A^2 P_\alpha \mathscr{D} = -\omega k Q_\alpha \mathscr{C}, \\ \Gamma Q_\alpha\{(k^2 M_\alpha - N_\alpha)\alpha - k_A^2 L_\alpha(1 - i\delta_\perp \alpha)\}\mathscr{B} &= -k\Omega M_\alpha \alpha P_\alpha \mathscr{D}, \\ \Gamma k\Omega Q_\alpha S_\alpha \mathscr{B} &= M_\alpha\{k^2 Q_\alpha - k^2 P_\alpha(1 + i\delta' k^2 - i\delta_\parallel \alpha)\}\mathscr{D}, \end{aligned}\right\} \tag{31}$$

which follow on substituting (30) into (19), (22), (24) and (25). Thus only one of the four constants introduced in (30) is independent. The dispersion relation is the condition that (31) have no zero values for these constants.

The axial velocity of the neutral gas velocity can be computed as follows. From the axial component of (15), the divergence of (15) and equation (20) we find that

$$\{M(Q-ia\nabla^2)-iaN\}\,v_{1z} = M(Q-ia\nabla^2)\,Qv_{n1z}, \tag{32}$$

and

$$k\{M(Q-ia\nabla^2)-iaN\}\,\nabla\,.\,\mathbf{v}_{n1} = iQNv_{n1z}. \tag{33}$$

Hence if v_{n1z} is expressed in the same form as adopted for v_{1z} in (30), we find from (32) that

$$\mathscr{B}_n = \mathscr{B}\{R_\alpha+\tfrac{1}{3}i\gamma'(Q_\alpha-4ia\alpha-iak^2)\}/M_\alpha, \tag{34}$$

where $\mathscr{B}_n$ relates to the neutral gas. The curl of (15) gives $\boldsymbol{\zeta}_1 = Q\boldsymbol{\zeta}_{n1}$, and hence the coefficient $\mathscr{D}_n$ for the neutral gas, corresponding to $\mathscr{D}$ in (30), is

$$\mathscr{D}_n = \mathscr{D}/Q_\alpha. \tag{35}$$

5. Calculation of transverse components

Once the solutions for the axial components of $\mathbf{B}_1$, $\mathbf{j}_1$, $\mathbf{v}_1$ and $\boldsymbol{\zeta}_1$ are known, the transverse components, i.e. the components in the radial and azimuthal directions, are easily derived. The theory is as follows.

We have $\nabla\,.\,\mathbf{B}_1 = 0$ and $\mathbf{n}\,.\,\nabla\times\mathbf{B}_1 = \mu j_{1z}$, i.e.

$$\frac{1}{r}\frac{d}{dr}(rB_{1r})+\frac{im}{r}B_{1\theta} = -ikB_{1z},\quad \frac{1}{r}\frac{d}{dr}(rB_{1\theta})-\frac{im}{r}B_{1r} = \mu j_{1z}.$$

Thus

$$\frac{1}{r}\frac{d}{dr}\{r(B_{1r}+iB_{1\theta})\}+\frac{m}{r}(B_{1r}+iB_{1\theta}) = i(\mu j_{1z}-kB_{1z}),$$

so

$$B_{1r}+iB_{1\theta} = ir^{-(m+1)}\int r^{m+1}(\mu j_{1z}-kB_{1z})\,dr.$$

From (30) and the result $\int x^{m+1}J_m(x)\,dx = x^{m+1}J_{m+1}(x)$ it now follows that

$$B_{1r}+iB_{1\theta} = i\mathscr{C}J_{m+1}(k_cr)-ik\mathscr{A}J_{m+1}(k_cr).$$

Similarly

$$B_{1r}-iB_{1\theta} = i\mathscr{C}J_{m-1}(k_cr)+ik\mathscr{A}J_{m-1}(k_cr).$$

On adding and subtracting these results and using the relations

$$(2m/x)\,J_m(x) = J_{m-1}(x)+J_{m+1}(x),\quad 2J'_m(x) = J_{m-1}(x)-J_{m+1}(x),$$

we get

$$B_{1r} = (i\mathscr{C}m/k_cr)\,J_m(k_cr)+ik\mathscr{A}J'_m(k_cr),\quad B_{1\theta} = -\mathscr{C}J'_m(k_cr)-(km/k_cr)\,\mathscr{A}J_m(k_cr). \tag{36}$$

With $\nabla\,.\,\mathbf{v}_1$ calculated from (20), we find from a derivation similar to that just given that

$$\left.\begin{aligned} v_{1r} &= i\frac{\mathscr{D}m}{k_cr}J_m(k_cr)+\frac{i}{k}\left(k^2-\frac{N_\alpha}{M_\alpha}\right)\Gamma\mathscr{B}J'_m(k_cr),\\ v_{1\theta} &= -\mathscr{D}J'_m(k_cr)-\frac{m}{k_crk}\left(k^2-\frac{N_\alpha}{M_\alpha}\right)\Gamma\mathscr{B}J_m(k_cr).\end{aligned}\right\} \tag{37}$$

From (36) and $\nabla \times \mathbf{B}_1 = \mu \mathbf{j}_1$, we get

$$\left.\begin{aligned}\mu j_{1r} &= \frac{imk_c}{r}\mathscr{A}\left(1+\frac{k^2}{k_c^2}\right)J_m(k_c r)+ik\mathscr{C}J_m'(k_c r),\\ \mu j_{1\theta} &= -\frac{mk}{k_c r}\mathscr{C}J_m(k_c r)-k_c^2\left(1+\frac{k^2}{k_c^2}\right)\mathscr{A}J_m'(k_c r),\end{aligned}\right\} \tag{38}$$

and by a similar method we could calculate ζ_{1r} and $\zeta_{1\theta}$ from (37), but these will not be required below.

We can calculate the electric field as follows. The axial component of (13) gives

$$E_{1z} = \sigma_{\parallel}^{-1} j_{1z} - iB_0\Omega(1+i\lambda)\,v_{1z} - B_0\lambda\Omega v_{n1z} + (B_0/\rho_0\omega_{ci})\,\mathbf{n}.\nabla.\mathbf{p}_{i1}, \tag{39}$$

and the last term of this equation can be expressed in terms of v_{1z} and g as indicated at the beginning of § 4. Equations (30) and (34) now permit us to express the right-hand side of (39) in terms of the known solutions for j_{1z} and v_{1z}. This is a very complicated equation but fortunately it is not required below. The transverse components of $\mathbf{E}_1$ then follow from $\nabla \times \mathbf{E}_1 = i\omega \mathbf{B}_1$, which gives

$$E_{1\theta} = -\frac{\omega}{k}B_{1r}+\frac{m}{rk}E_{1z}, \quad E_{1r} = \frac{\omega}{k}B_{1\theta}-\frac{i}{k}\frac{dE_{1z}}{dr}. \tag{40}$$

6. The field external to the plasma

In order to apply boundary conditions to the plasma it is necessary to calculate the magnetic and electric fields outside the plasma cylinder. We shall assume no displacement currents and a conductivity σ_e in the material or space external to the plasma. In this case equations (11) to (15) reduce to

$$\nabla \times \mathbf{B}_1^e = \mu\sigma_e \mathbf{E}_1^e, \quad \nabla \times \mathbf{E}_1^e = i\omega \mathbf{B}_1^e,$$

the superscript e denoting external field values. Thus

$$(\nabla^2 + i\omega\mu\sigma_e)\,\mathbf{B}_1^e = 0,$$

and on solving this by a method similar to that used for $\nabla^2\chi = \alpha\chi$, we find

$$\left.\begin{aligned}B_{1r}^e &= -ik\mathscr{F}K_m'(\kappa r)-i\mu\sigma_e(m/\kappa r)\,\mathscr{G}K_m(\kappa r),\\ B_{1\theta}^e &= (km/\kappa r)\,\mathscr{F}K_m(\kappa r)+\mu\sigma_e\mathscr{G}K_m'(\kappa r),\\ B_{1z}^e &= \kappa\mathscr{F}K_m(\kappa r),\end{aligned}\right\} \tag{41}$$

where $\kappa^2 = k^2 - i\omega\mu\sigma_e$, and $\mathscr{F}$, $\mathscr{G}$ are constants to be determined by the boundary conditions. The axial component of the electrical field is found to be

$$E_{1z}^e = \kappa\mathscr{G}K_m(\kappa r), \tag{42}$$

then the transverse components can be written down by an application of (40).

If the plasma is surrounded by an infinitely conducting wall, then the electric field $\mathbf{E}_1^e$ must be zero. It follows that $\mathbf{B}_1^e = 0$, and we have the special case of (41) in which

$$\mathscr{F} = \mathscr{G} = 0. \tag{43}$$

7. The boundary conditions

For a plasma of given physical characteristics, (27) and (28) yield eight dispersion relations, say

$$f_i(k, \omega, k_{ci}) = 0 \quad (i = 1, 2, \ldots, 8), \tag{44}$$

in which we are free to choose only one of the k_{ci} to satisfy the boundary conditions; suppose this is k_{c1}, then, for a given ω, k follows from $f_1 = 0$, and then $k_{c2}, \ldots, k_{c8}$ follow from the remaining seven equations of (44). At least one of the constants A_i in (29) will depend on the initial conditions, so that in general the seven ratios of these constants have also to be assigned. Thus adding to these ratios and k_{c1} the two constants appearing in (41), we arrive at a total of ten parameters so far undetermined. In general these parameters will be complex numbers.

Let P_m be the degree of (27) in α ($P_m = 8$ in the general case, but less in special cases), then P_m is the number of propagation modes, and if the number of boundary conditions is N_b, a unique wave can be transmitted along the tube only if

$$N_b = P_m + 2. \tag{45}$$

If N_b exceeds this value no wave is possible.

Turning now to the boundary conditions, we shall assume for the moment that on the tube wall, $r = r_0$, there is a current sheet vector $\mathbf{j}_1^*$ and an electric dipole layer of strength τ_1. Then integration of (2) to (5) over a thin boundary layer on the tube wall leads to the following boundary conditions for the perturbations:

$$[0, B_{1\theta}, B_{1z}] = \mu(0, -j_{1z}^*, j_{1\theta}^*), \tag{46}$$

$$[0, E_{1\theta}, E_{1z}] = -\left(0, \frac{im}{r_0}\tau_1, ik\tau_1\right)\Big/\epsilon_0, \tag{47}$$

$$\Gamma v_{1r} = 0, \quad \Gamma_n v_{n1r} = 0, \tag{48}$$

$$\gamma_\perp v_{1\theta} = 0, \quad \gamma_\parallel v_{1z} = 0, \quad \gamma_n v_{n1\theta} = 0, \quad \gamma_n v_{n1z} = 0, \tag{49}$$

where $[X]$ denotes the jump in X across the boundary. (For the theory of double-layer distributions see Stratton 1941, p. 188.) Equation (5) also yields the continuity of the sum of the material and magnetic pressures across the boundary, but at a solid boundary these conditions can be dropped as the boundary itself provides any required balancing stresses. From the first of (40) we see that (47) is equivalent to

$$[B_{1r}] = 0 \tag{50}$$

and

$$[E_{1\theta}] = -(im/r_0\epsilon_0)\,\tau_1. \tag{51}$$

Thus if $\mathbf{j}_1^*$ and τ_1 are zero, (46) and (48) to (51) provide ten boundary conditions, i.e. $N_b = 10$, and waves composed of less than eight modes cannot be transmitted.

With highly conducting walls the conditions in (43) must be added, which would increase N_b by two. However, in this case a current sheet is likely to occur, when (46) imposes no restriction, but just serves to define $\mathbf{j}_1^*$, and N_b remains unchanged.

If the gases are assumed to be inviscid the four conditions in (49) can be dropped. The first condition in (48) can be neglected if the parameter β, defined by

$$\beta = \tfrac{1}{2}\gamma(2\mu p_0/B_0^2) = (\gamma p_0/\rho_0)(\mu\rho_0/B_0^2) = C^2/v_A^2 = \Gamma k_A^2, \qquad (52)$$

(here γ is the ratio of the specific heats) is small. Then on assuming that C_n is not large compared with C, the second of (48) can also be neglected. The following table sets out some important special cases.

Case	G	a	ab	ac	abc	abcd	ace	abce	abcde
P_m	8	6	5	4	3	2	3	2	1
N_b	10	8	7	6	5	4	6	5	4

a = zero neutral gas viscosity, d = zero ionized gas pressure.
b = zero neutral gas pressure, e = zero resistivity,
c = zero ionized gas viscosity, G = general case,

Here N_b is calculated assuming that $\tau_1 = 0$, and that there is a current sheet on highly conducting walls. Notice that (45) is satisfied except for the cases involving zero resistivity; this is because the absence of resistivity does not reduce the number of boundary conditions, at least in an obvious way (but see footnote on p. 580).

Now the experimental conditions usually achieved are close to the condition abcde; pressure and resistivity appear as second-order effects, modifying the wave velocity and attenuation rather than providing new modes, and it seems unreasonable to evoke second-order modes in order to satisfy (45). However, the experimental evidence is that hydromagnetic waves can be propagated along both insulating and conducting tubes. Fortunately there is another method of satisfying (45), namely, that of postulating the presence of an electric dipole layer on the wall. This hypothesis is physically reasonable as plasmas do tend to shield themselves from external electric fields (or their absence) by surface charge separation. Accepting this we can discard (51) as an effective boundary condition, and we are then left with (for the case abcde)

$$[B_{1r}, B_{1\theta}, B_{1z}] = \mu(0, -j_{1z}^*, j_{1\theta}^*).$$

Unfortunately this fails with insulating walls, for if $\sigma_e = 0$, (40) to (42) show that $\mathbf{B}_e^1$ depends only on $\mathscr{F}$, and there are therefore only two constants available to satisfy the three conditions just given (in which $\mathbf{j}_1^* = 0$). We shall return to this point below.

With highly conducting walls $\mathbf{j}_1^* \neq 0$, and only the first condition is effective. From (36), (41) and (43) the condition in this case can be written

$$B_{1r} = (i\mathscr{C}m/k_c r_0)J_m(k_c r_0) + ik\mathscr{A}J_m'(k_c r_0) = 0. \qquad (53)$$

If we make the further restriction that the ion-cyclotron effect is negligible, i.e. $\Omega \approx 0$, it follows from (39), (42) and (43) that both E_{1z} and E_{1z}^e are zero, so that (51) is satisfied *without* a dipole layer being required. On the other hand when $\Omega \neq 0$, these equations show that $E_{1z} \neq 0$, so that there must be a discontinuity in the tangential electric field at the wall. This connexion between the presence of a dipole layer and the ion-cyclotron effect arises because the differing Larmor radii of the ions and electrons inevitably leads to some charge separation at the boundary.

In other cases $\mathbf{j}_1^*$ is zero† and our boundary condition becomes $B_{1r} = B_{1r}^e$, $B_{1\theta} = B_{1\theta}^e$, $B_{1z} = B_{1z}^e$, and on eliminating $\mathscr{G}$ and $\mathscr{F}$ from these and (41) we arrive at a single relation between B_{1r}, $B_{1\theta}$ and B_{1z}. As $\mu j_{1r} = (im/r)B_{1z} - ikB_{1\theta}$, this boundary relation can be written

$$m\mu j_{1r} = \kappa^2 r_0 \chi(B_{1r} + i\chi B_{1z}) + (m^2\omega\mu\sigma_e/r_0^2\kappa^2) = \mathscr{H}\sigma_e, \quad \text{at} \quad r = r_0, \tag{54}$$

where $\mathscr{H}$ is a constant, and

$$\chi \equiv (k/\kappa)\, K_m'(\kappa r_0)/K_m(\kappa r_0). \tag{55}$$

From (30), (36), (38) and (54) we can write down a boundary relation like (53) for the case when σ_e is finite and non-zero. However, if the walls are insulating, $\sigma_e = 0$, and (54) *splits into the two boundary relations*

$$\mu j_{1r} = i((m/r_0)B_{1z} - kB_{1\theta}) = 0, \quad B_{1r} + i\chi_0 B_{1z} = 0, \tag{56}$$

where $\chi_0 \equiv K_m'(kr_0)/K_m(kr_0)$. As these cannot both be satisfied by a single value of k_c, we conclude that, in general, with insulating walls a pure mode cannot be propagated, unless a current sheet forms at the wall (see footnote below). However as described later, there are special cases in which only one of (56) need be satisfied.

8. Negligible viscosity and pressure

This special case, which corresponds to the experimental conditions, will now be considered further. Put $\gamma_\parallel$, $\gamma_\perp$, γ_n, Γ_n and Γ zero in (16), (18) and (21) and there results

$$\left.\begin{aligned} a = 0, \quad R_\alpha = 0, \quad M_\alpha = 0, \quad P_\alpha = 1 + \lambda\xi - i\xi, \\ L_\alpha = P_\alpha^2, \quad N_\alpha = sP_\alpha^2, \quad S_\alpha = P_\alpha^2, \\ \frac{\Gamma N_\alpha}{M_\alpha} = \frac{sP_\alpha^2}{s^2P_\alpha^2 + b}, \quad \frac{M_\alpha}{\Gamma L_\alpha} = \frac{s^2P_\alpha^2 + b}{P_\alpha^2}, \end{aligned}\right\} \tag{57}$$

where
$$b = \frac{C_n^2 \rho_{n0}}{C^2 \rho_0}, \tag{58}$$

$$s = \frac{Q_\alpha}{P_\alpha} = \frac{1 - i\xi}{1 + \lambda\xi - i\xi} = \frac{\lambda^2\mathscr{I} + (1-\mathscr{I})^2}{\lambda^2 + (1-\mathscr{I})^2} - i\frac{\lambda(1-\mathscr{I})^2}{\lambda^2 + (1-\mathscr{I})^2}, \tag{59}$$

and $\mathscr{I} \equiv \rho_0/(\rho_0 + \rho_{n0})$ is the degree of ionization. With these values (27) reduces to the quadratic in k^2,

$$\{sk^2 - k_A^2(1 + i\delta_\perp k^2 + i\delta_\parallel k_c^2)\}\{s(k^2 + k_c^2) - k_A^2[1 + i\delta_\perp(k^2 + k_c^2)]\} = k^2(k^2 + k_c^2)\,\Omega^2. \tag{60}$$

† There is always the possibility that a consequence of small resistivity is the presence of a current sheet on the edge of the plasma *regardless* of the nature of the walls. If this is so, (53) is the boundary condition in all cases, and the distinction between insulating and conducting walls vanishes. This seems rather unlikely in experiments because one expects the conductivity to be rather lower in the gas adjacent to a cold wall. This point can only be resolved by experiment; here we shall pursue the consequences of there being no current sheet on insulating walls, as the other case is covered in the highly-conducting-wall treatment.

Let s_1, $-s_2$ denote the real and imaginary parts of s as given in (59) and write

$$\left.\begin{aligned} h &\equiv s_1^2-\Omega^2, \quad g \equiv k_A^2 - s_1 k_c^2, \quad f_2 \equiv s_2+\delta_\perp k_A^2, \\ A &\equiv h - f_2^2, \quad B_2 \equiv -2gf_2 + s_1\delta_\parallel k_c^2 k_A^2, \\ B_1 &\equiv 2s_1 k_n^2 - hk_c^2 + k_c^2 f_2(f_2+\delta_\parallel k_A^2), \quad A_2 \equiv 2s_1 f_2, \\ C &\equiv k_A^2 g - \delta_\parallel f_2 k_A^2 k_c^4, \quad C_2 \equiv k_A^2 k_c^2(\delta_\parallel g + f_2), \end{aligned}\right\} \tag{61}$$

then the quadratic can be written

$$(A-iA_2)k^4-(B_1+iB_2)k^2+(C+iC_2) = 0, \tag{62}$$

which has the solution

$$k^2 = \tfrac{1}{2}\{(A+iA_2)/(A^2+A_2^2)\}\{B_1+iB_2 \pm (G_1+iG_2)^{\frac{1}{2}}\}, \tag{63}$$

where $G_1 \equiv B_1^2-4AC-4A_2C_2-B_2^2$ and $G_2 \equiv 2B_1B_2-4AC_2+4A_2C$.

Let
$$k = \eta + i\epsilon, \tag{64}$$

then the z-dependence of the waves becomes $\exp(i\eta z - \epsilon z)$ so that $\eta/2\pi$ is the number of waves per unit length, and ϵ is the absorption coefficient. The phase velocity of the waves is

$$v_p = \omega/\eta. \tag{65}$$

In a Culham Laboratory report to be issued soon the numerical solution of (63) for v_p and ϵ will be discussed; here we shall just note some of the salient features, restricting our attention to the case when the damping caused by resistivity and neutrals are small. In this case the subscripts 1 and 2 in (61) will denote the order of magnitude of the labelled quantities. The orders of A and C are not indicated because they depend on the numbers h and g, which do vanish at certain critical frequencies.

If A is first order and positive, then correct to second order the above theory yields

$$\eta^2 = (1/2h)\{2s_1k_A^2 - hk_c^2 \pm (k_c^4h^2+4k_A^4\Omega^2)^{\frac{1}{2}}\}, \tag{66}$$

$$\epsilon = \frac{k_A^2}{2\eta h^2}\left\{(2s_1^2-h)f_2 + \tfrac{1}{2}hs_1\delta_\parallel k_c^2 \pm \frac{8s_1k_A^2\Omega^2 f_2 + h\delta_\parallel k_c^2(hk_c^2 s_1 + 2\Omega^2 k_A^2)}{2(k_c^4h^2+4k_A^4\Omega^2)^{\frac{1}{2}}}\right\}. \tag{67}$$

From (65) and (66) it is clear that the positive sign gives a slower wave then does the negative sign, and we shall therefore distinguish the two solutions as being the 'slow' (+) and 'fast' (−) waves. At low frequencies $h \approx s_1^2$, and the equations reduce to

$$\left.\eta^2 = \frac{k_A^2}{s_1} - \begin{cases} 0 & \text{(slow)}, \\ k_c^2 & \text{(fast)}; \end{cases} \qquad \epsilon = \frac{k_A^2}{2s_1^2\eta}\left(f + \begin{matrix}\delta_\parallel s_1 k_c^2 \\ 0\end{matrix}\right) \begin{matrix}\text{(slow)}, \\ \text{(fast)}.\end{matrix}\right\} \tag{68}$$

If either A or C is small, it follows from (62) that

$$k^2 = B_1/(A-iA_2) \quad \text{(slow wave)}, \qquad k^2 = (C+C_2)/B_1 \quad \text{(fast wave)}. \tag{69}$$

Suppose A is small, then for the slow wave

$$2\eta^2 = \{B_1/(A^2+A_2^2)\}\{(A^2+A_2^2)^{\frac{1}{2}}+A\}, \quad 2\epsilon^2 = \{B_1/(A^2+A_2^2)\}\{(A^2+A_2^2)^{\frac{1}{2}}-A\}.$$

From these, and neglecting second-order terms, we find that ϵ has a maximum value of

$$\left.\begin{aligned} \epsilon_{\max} &= \sqrt{(3\sqrt{3}/8)}\,k_A/\sqrt{f_2} \quad \text{at} \quad A = -A_2/\sqrt{3}, \\ \text{i.e. at} \quad \omega_{\text{crit}}^2 &= s_1\{s_1+(2/\sqrt{3})f_2\}\,\omega_{ci}^2. \end{aligned}\right\} \tag{70}$$

Similarly η has a maximum, and hence v_p a minimum,

$$(v_p)_{\min} = \sqrt{(8/3\sqrt{3})}\sqrt{f_2}\,v_A \quad \text{at} \quad \omega^2_{\text{crit}} = s_1\{s_1-(2/\sqrt{3})f_2\}\,\omega^2_{ci}. \tag{71}$$

If there is no damping the absorption becomes infinite, and the slow wave cannot be propagated at frequencies above ω_{crit}. There is no resonance absorption effect on the fast wave at these critical frequencies.

Now suppose C is small, then for the fast wave

$$\eta^2 = \tfrac{1}{2}\{(C^2+C_2^2)^{\frac{1}{2}}+C\}/B_1, \quad \epsilon^2 = \tfrac{1}{2}\{(C^2+C_2^2)^{\frac{1}{2}}-C\}/B_1.$$

At $C = 0$,

$$\eta^2 = k_A^2(f_2+\delta_\parallel g)/(s_1^2+\Omega^2),$$

and the wave-number is very small. Without damping, the fast wave is cut off sharply at $C = k_A^2 g = 0$, i.e. at $\omega_{\text{crit}} = \sqrt{s_1}\,k_c v_A$. Damping reduces the sharpness of the cut-off frequency.

9. Calculation of k_c

To calculate k_c we need to use the boundary conditions. Now (31) and (57) yield

$$\begin{aligned}\mathscr{C}/\mathscr{A} &= -\{s(k^2+k_c^2)-k_A^2[1+i\delta_\perp(k^2+k_c^2)]\}/k\Omega \\ &= -k(k^2+k_c^2)\,\Omega/\{sk^2-k_A^2(1+i\delta_\perp k^2+i\delta_\parallel k_c^2)\}.\end{aligned} \tag{72}$$

First, suppose the walls are highly conducting, then (53) holds, and if $\mathscr{C}/\mathscr{A}$ is eliminated from (53) and (72) one arrives at a complicated equation determining k_c. On dropping second-order quantities we get

$$k_c r_0 \frac{J'_m(k_c r_0)}{J_m(k_c r_0)} = m\Omega\,\frac{\eta^2+k_c^2}{s_1\eta^2-k_A^2} = m\,\frac{s_1(\eta^2+k_c^2)-k_A^2}{\eta^2\Omega}. \tag{73}$$

To complete the solution it remains only to use (72) to eliminate the ratio $\mathscr{C}/\mathscr{A}$ from (36) and (38) (see equations (80) below).

Notice from (68) and (73) that when Ω is small we must use the first right-hand side in (73) for the fast wave, and the second for the slow wave. Thus when Ω is negligible (73) reduces to

$$m \neq 0 \quad J'_m(k_c r_0) = 0 \quad \text{(fast wave)}, \qquad J_m(k_c r_0) = 0 \quad \text{(slow wave)}; \tag{74}$$

$$m = 0 \quad J'_0(k_c r_0) = 0 \quad \text{(both waves)}. \tag{75}$$

The effect of a small ion-cyclotron term on the fast wave cut-off frequency can be computed as follows. On eliminating η^2 from the first of (68) and (73) we get

$$\frac{k_c r_0}{m}\frac{J'_m(k_c r_0)}{J_m(k_c r_0)} = -\frac{k_A^2}{s_1^2 k_c^2}\,\Omega. \tag{76}$$

At $\Omega = 0$ this reduces to $J'_m(k_c r_0) = 0$, which has a root k_{c1}, say. Let $k_c = k_{c1}+\Delta k_c$ be the corresponding root of (76), then to first order

$$\Delta k_c = -\frac{m}{r_0^2 k_{c1}}\frac{k_A^2}{s_1^2 k_{c1}^2}\frac{J_m(k_{c1}r_0)}{J''_m(k_{c1}r_0)}\,\Omega, \tag{77}$$

and the cut-off frequency $\sqrt{s_1}\,k_c v_A$ is reduced by an amount $\sqrt{s_1}\,\Delta k_c v_A$. This is another form of a result due to Newcombe (1957) for fully ionized plasmas.

The theory for the case when the walls are poor conductors is much the same as above except that (73) is replaced by a more complicated relation which follows from (36), (38), (54) and (72). With a perfect insulator the two relations in (56) have to be satisfied. Now if we can find perturbations such that either (i) j_{1r} is negligibly small *throughout* the field ($j_{1r} \sim 0$), or (ii) $B_{1r}, B_{1z} \sim 0$, then with (i) the second of (56) is the dominating boundary condition, and the first can be ignored, while with (ii) the first of (56) is dominant and the second can be ignored. By examining equations (80) and (81) below we see that these two cases can occur only if $m = 0$,† and if in addition:

Case (i) $\eta(\eta^2+k_c^2)\,\Omega \ll k_c(s_1\eta^2-k_A^2)$ (fast radial wave),

with boundary condition $$B_{1r}+i\chi_0 B_{1z} = 0; \tag{78}$$

Case (ii) $\eta^2\Omega,\ \eta k_c\Omega \ll s_1\eta^2-g$ (slow torsional wave),

with boundary condition $$B_{1\theta} = 0. \tag{79}$$

It appears unlikely that perturbations intermediate in character can be propagated with insulating walls, unless of course these walls behave in effect like conducting walls because of the presence of a current sheet.

In the experimental work it is the magnetic field strengths that are measured and so for reference we give here the values of $B_{1r}, B_{1\theta}, B_{1z}$ obtained by eliminating $\mathscr{C}/\mathscr{A}$ from (36) and omitting second-order terms. For slow waves it is convenient to write

$$\left.\begin{aligned} B_{1r} &= i\mathscr{C}\left\{\frac{m}{k_c r}J_m(k_c r)-\frac{\eta^2\Omega}{s_1\eta^2-g}J'_m(k_c r)\right\},\\ B_{1\theta} &= -\mathscr{C}\left\{J'_m(k_c r)-\frac{\eta^2 m}{k_c r}\frac{\Omega}{s_1\eta^2-g}J_m(k_c r)\right\},\\ B_{1z} &= -\mathscr{C}\frac{\eta\Omega}{s_1\eta^2-g}k_c J_m(k_c r),\end{aligned}\right\} \tag{80}$$

while for fast waves

$$\left.\begin{aligned} B_{1r} &= -i\mathscr{A}\left\{\frac{m}{k_c r}\frac{\eta(\eta^2+k_c^2)}{s_1\eta^2-k_A^2}\Omega J_m(k_c r)-\eta J'_m(k_c r)\right\},\\ B_{1\theta} &= \mathscr{A}\left\{\frac{\eta(\eta^2+k_c^2)}{s_1\eta^2-k_A^2}\Omega J'_m(k_c r)-\frac{\eta m}{k_c r}J_m(k_c r)\right\},\\ B_{1z} &= k_c\mathscr{A}J_m(k_c r).\end{aligned}\right\} \tag{81}$$

10. The special case $m = 0$

The experimental results so far obtained in the Culham Laboratory are for waves that are predominantly the $m = 0$ mode, and it is therefore useful to list the equations for this case.

† We exclude combinations of waves having equal and opposite values of m because (see (77)) such waves have different phase velocities.

From (80) and (81),

$$\left.\begin{aligned} B_{1r} &= i\mathscr{A}\eta J_0'(k_c r) = -i\mathscr{C}\{\eta^2\Omega/(s_1\eta-g)\}J_0'(k_c r),\\ B_{1\theta} &= \mathscr{A}\{\eta(\eta^2+k_c^2)/(s_1\eta^2-k_A^2)\}\,\Omega J_0'(k_c r) = -\mathscr{C}J_0'(k_c r),\\ B_{1z} &= k_c\mathscr{A}J_0(k_c r) = -\mathscr{C}\{\eta\Omega/(s_1\eta^2-g)\}k_c J_0(k_c r),\end{aligned}\right\} \quad (82)$$

from which we observe that at low frequencies the slow wave is almost entirely a torsional one. However, near the critical frequency given by (71), η is large, $\Omega \approx s_1$, and the ratio of the amplitude of B_{1r} to $B_{1\theta}$ tends to unity; B_{1z} remains small.

Similarly equations (38) give

$$\left.\begin{aligned} \mu j_{1r} &= -i\mathscr{A}\{\eta^2(\eta^2+k_c^2)/(s_1\eta^2-k_A^2)\}\,\Omega J_0'(k_c r) = i\eta\mathscr{C}J_0'(k_c r),\\ \mu j_{1\theta} &= -(\eta^2+k_c^2)\mathscr{A}J_0'(k_c r) = \mathscr{C}\{\eta(k_c^2+\eta^2)\,\Omega/(s_1\eta^2-g)\}J_0'(k_c r),\\ \mu j_{1z} &= -\mathscr{A}\{k_c\eta(\eta^2+k_c^2)/(s_1\eta^2-k_A^2)\}\,\Omega J_0(k_c r) = \mathscr{C}k_c J_0(k_c r).\end{aligned}\right\} \quad (83)$$

The boundary conditions are

$$\left.\begin{aligned} J_0'(k_c r_0) &= 0 \quad \begin{pmatrix}\text{both waves, conducting walls}\\ \text{torsional wave, insulating walls}\end{pmatrix},\\ \eta J_0'(k_c r_0)+k_c\chi_0 J_0(k_c r_0) &= 0 \quad \text{(radial wave, insulating walls).}\end{aligned}\right\} \quad (84)$$

11. The effect of viscosity

At very high temperatures viscosity contributes significantly to the wave damping. Such temperatures have not been achieved in the experiments carried out so far, but as it is hoped that hydromagnetic waves may have some value as a diagnostic technique for really hot plasmas, it is worth noting here the first-order effects of viscosity. The theory of § 8 can be modified as follows.

If the viscosity terms are retained as small, second-order terms, but the pressure terms are neglected, equation (27) becomes, correct to third-order terms,

$$(k^2s_1-k_A^2-ik^2f_2'-ik_A^2k_c^2\delta_\parallel)\{k^2s_1-g-i(k^2+k_c^2)f_2'-i\phi_2\} = k^2(k^2+k_c^2)\,\Omega^2(1-i\psi_2), \quad (85)$$

where

$$\left.\begin{aligned} f_2' &= f_2+s_1(k^2+k_0^2)\{s_1\gamma_\perp+(1-s_1)\gamma_n\},\\ \phi_2 &= s_1\gamma_\perp\{s_1(k^2+k_c^2)^2-k_A^2(k^2+\tfrac{2}{3}k_c^2)\}-s_1\gamma_\parallel(k^2s_1-g)(\tfrac{4}{3}k^2+k_c^2)\\ &\quad +\tfrac{1}{3}\gamma_n(k^2+k_c^2)\{(1-s_1)(k_A^2-s_1k^2)-[s_1\xi^2/(1+\xi^2)](k^2s_1-g)\},\\ \psi_2 &= s_1\gamma_\perp(k^2+\tfrac{2}{3}k_c^2)-s_1\gamma_\parallel(\tfrac{4}{3}k^2+k_c^2).\end{aligned}\right\} \quad (86)$$

Comparing this with (60) and (61), which are exact for zero viscosity, we see that the order of the dispersion equation is considerably raised by the presence of viscosity. However, if the new terms are treated as small perturbations, we can ignore this change in order, and simply calculate the effects of these perturbations on the roots of the original quadratic. From (85) we find that the coefficients in (61) are thus modified to

$$\left.\begin{aligned} A &= h, & A_2 &= 2s_1f_2'-\Omega^2\psi_2,\\ B_1 &= 2s_1k_A^2-hk_c^2, & B_2 &= -2gf_2'+s_1\delta_\parallel k_c^2k_A^2+s_1\phi_2-k^2\Omega^2\psi_2,\\ C &= k_A^2g, & C_2 &= k_A^2k_c^2(\delta_\parallel g+f_2')+k_A^2\phi_2,\end{aligned}\right\} \quad (87)$$

where, as we are neglecting fourth-order terms, only the dominant terms in A, B_1 and C have been retained. With this modification the rest of the theory of § 8 is easily adjusted to allow for viscosity.

Consider, for example, the damping of the slow and fast waves at frequencies well below the critical values $s_1\omega_{ci}$. When the viscosity is zero, the second of (68) gives the damping or absorption coefficient. First, take the slow wave. The appropriate dispersion relation is obtained by equating the first factor in (85) to zero. To first order, $k^2 = k_A^2/s_1$, and using this to eliminate k^2 from the second-order terms, we arrive at

$$\epsilon = (k_A^2/2s_1^2\eta)\{f_2+\delta_{\|}s_1k_c^2+(k_A^2+s_1k_c^2)[s_1\gamma_{\perp}+(1-s_1)\gamma_n]\}. \tag{88}$$

Similarly from the second factor of (85) it follows that for the fast wave

$$\epsilon = (k_A^2/2s_1^2\eta)\{f_2+(k_A^2+\tfrac{1}{3}s_1k_c^2)[s_1\gamma_{\perp}+(1-s_1)\gamma_n]\}. \tag{89}$$

It is interesting to note that for these waves the viscosity factor $\gamma_{\|}$ is absent, i.e. the viscosity involved is that which damps motions wholly normal to the magnetic field ($\nu_{\perp}$). The other viscosity $\nu_{\|}$, which is considerably smaller than $\nu_{\perp}$ due to the influence of the magnetic field, will be effective only at relatively high values of Ω.

12. The effect of gas pressure

Suppose for simplicity that we have a plasma in which the resistivity and viscosity are negligible, but the pressures are not. In this case the dispersion equation (27) reduces exactly to

$$(k^2s-k_A^2)\left\{k_A^2+\alpha s-\alpha s(\alpha+k^2)\frac{s\Gamma_m+i\xi(\Gamma_n-s\Gamma-s\Gamma_n-s\alpha\Gamma\Gamma_n)}{1+s\alpha\Gamma_m-i\xi[1+s\alpha(\Gamma+\Gamma_n+\alpha\Gamma\Gamma_n)]}\right\} = \alpha k^2\Omega^2, \tag{90}$$

where

$$\Gamma_m \equiv \Gamma+\lambda\xi\Gamma_n = \rho_0^{-1}(\rho_0\Gamma+\rho_{n0}\Gamma_n). \tag{91}$$

This is a cubic in α, and as $-\alpha = k^2+k_c^2$, to each of the three roots of (90) there correspond at least a pair of waves moving at different speeds, i.e. at least six distinct waves in all. As the gas pressure falls, Γ_m, Γ and Γ_n tend to zero, and (90) becomes a linear equation in α, the single root of which yields the pair of waves ('fast' and 'slow') discussed in § 8.

If the gas is fully ionized, $s = 1$, $\xi = 0$, and (90) reduces to the quadratic

$$k^2\Omega^2\Gamma\alpha^2+\{k^2\Omega^2+(k_A^2-k^2)+\Gamma(k_A^2-k^2)^2\}\alpha+k_A^2(k_A^2-k^2) = 0, \tag{92}$$

and it is readily verified that if $\Gamma k_A^2 = \beta$ (see equation (52)) is small, one root of this yields the pair of hydromagnetic waves of § 8, while the other yields the pair of acoustic waves $k^2 = 1/\Gamma-k_c^2$, i.e. by (64) and (65),

$$v_p = \pm C\{1-k_c^2C^2/\omega^2\}^{-\frac{1}{2}}. \tag{93}$$

In general the roots of (92) yield complicated mixtures of hydromagnetic and acoustic waves.

Of more interest here is the way in which small but significant pressures affect the results of § 8. In this case we can ignore second-order terms like $\Gamma_n \Gamma$, $s_2 \Gamma$, etc., and reduce (90) by (59) to the form

$$\{k^2(s_1 - is_2) - k_A^2\}\{k^2(s_1' - is_2) - g' - ik_c^2 s_2\} = k^2(k^2 + k_c^2)\,\Omega^2, \tag{94}$$

with $\quad s_1' \equiv s_1(1 + k_c^2 \Gamma_a), \quad g' \equiv k_A^2 - s_1' k_c^2 \quad$ and $\quad \Gamma_a \equiv \Gamma + \{(1 - s_1)/s_1\}\,\Gamma_n. \qquad (95)$

Thus in the notation of (61) and (62) we now have the coefficients

$$A = s_1 s_1' - \Omega^2, \quad B_1 = (s_1 + s_1')\,k_A^2 - s_1 s' k_c^2 + \Omega^2 k_c^2, \quad C = k_A^2 g', \tag{96}$$

correct to third order, with the coefficients A_2, B_2, C_2 unchanged. Clearly we can linearly superimpose pressure and viscosity effects by combining the coefficients A, B_1 and C from (96) with A_2, B_2 and C_2 from (87).

Notice from the first factor of (94) that near $\Omega = 0$ the slow wave is unaffected by gas pressure, a result which is true for all values of the pressure (cf. (90)). The fast wave near $\Omega = 0$ is found from the second factor of (94) to have a wave-number

$$\eta = \frac{k_A^2}{s_1} - k_c^2\left(1 + \frac{\beta_a}{s_1}\right), \tag{97}$$

where (see (52)) $\qquad \beta_a = k_A^2 \Gamma_a = k_A^2\{\Gamma + [(1 - s_1)/s_1]\,\Gamma_n\}.$

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